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The stationary solution of a random dynamical model

Wei Li,[1, a)] Junfeng Zhao,[2, b)] Natasa Trisovic,[3, c)] Ruihong Li[1, d)]
[1)] *School of Mathematics and Statistics, Xidian University, Xi'an 710071, China*
[2)] *Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an 710072, China*
[3)] *Department of Mechanics, University of Belgrade, Belgrade 11120, Serbia*

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Abstract This paper studies the stationary probability density function (PDF) solution of a nonlinear business cycle model subjected to random shocks of Gaussian white-noise type. The PDF solution is controlled by a Fokker–Planck–Kolmogorov (FPK) equation, and we use exponential polynomial closure (EPC) method to derive an approximate solution for the FPK equation. Numerical results obtained from EPC method, better than those from Gaussian closure method, show good agreement with the probability distribution obtained with Monte Carlo simulation including the tail regions.

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Keywords Gaussian white-noise, business cycle, stationary solution

The study of business cycle (or called trade cycle) is a hot issue in the macroeconomic research, which is referred to the fluctuation in economic activities due to the change in the economic variables, such as employment, income, output, prices, etc.[1] The fluctuation,[2] is generally the most significant indicator to display the economic evolution over a short or long period of time, which usually can be casted as nonlinear dynamical models under random shocks if it is considered within the field of nonlinear science. Hence, the theoretical study on such dynamical business cycle models subjected to random shocks becomes an important work, which will be greatly helpful for us to better understand and predict the change of a certain economic variable.

Random shocks, such as instant accident in a company, government interventions on policies, natural disaster, critical change in global economic situation, unexpected wars, and so on, are definitely associated with the economic fluctuations, and generally can be formulated as a random process. For convenience, they are usually assumed to come from the same probability distribution,[3] typically a Gaussian white-noise.[4] Actually, Gaussian white-noise is a special random process with zero-mean and constant autocorrelation function. It is commonly regarded as the best approximation of many real-world random situations. Therefore, dynamical business cycle models with random shocks of Gaussian white-noise type are always a hot topic in the field of macroeconomics and random dynamics, and many valuable references of them have been achieved in recent years.[5–8]

It should be mentioned that we pay more attention to the problems about the long-run prediction for a certain economic variable in macroeconomics. Regarding these problems, we can turn

[a)] Corresponding author. Email: liweilw@mail.xidian.edu.cn.
[b)] Email: zhaojf@nwpu.edu.cn.
[c)] Email: ntrisovic@mas.bg.ac.rs.
[d)] Email: llylrh8077@126.com.

to explore the stationary response of the dynamical business cycle model subjected to random shocks, because the stationary response of a dynamical system exactly dedicates to the survey of the long-run evolution of system variables over the time. Now the key point of the problems is how to get the stationary response of the dynamical business cycle model. Over the past two decades, equivalent linearization procedure,[9] perturbation method,[10] and stochastic averaging method[11,12] are developed and often used to get approximate solutions of stationary response of a random dynamical system. However, there are always some restrictions associated with these methods on using them. Equivalent linearization procedure is regarded to be unsuitable to get the approximate solution when multiplicative random shocks are involved; perturbation method is limited in the case of weak nonlinear equations with a suited initial solution and a previously determined perturbation parameter. Comparatively, stochastic averaging method is more effective to solve the responses of dynamical systems with nonlinear and multiplicative random shocks. Nonetheless, it is only feasible to the systems with light damping and weak shocks.

Exponential polynomial closure (EPC), which is originally proposed by Er et al.,[13–15] is a new developing approximate method with high accuracy to get the approximate solution of a random dynamical system with no constraint on light damping and weak shocks. The main idea of this strategy is that the stationary responses of the random dynamical system characterized by probability density function (PDF) is approximated by an exponential polynomial function with unknown parameters in it. After that, in the weak sense of integration, special measures are applied to make it satisfy the governing Fokker–Planck (FP) equation. Next, the estimation of the unknown parameters in approximate PDF can be worked out by solving a series of algebraic equations.

In the present paper, we aim at utilizing the EPC strategy to determine the approximate PDF response solution for a dynamical business cycle model under Gaussian white-noise shocks.

We extend our preceding work[16] and continue to consider the business cycle model characterized by the combination ideas of Goodwin,[17] Puu and Sushko.[18] In addition, we model the random shocks on the real business cycle as two independent Gaussian white-noises. The corresponding dynamical model can be expressed in the framework of mathematics as a nonlinear differential equation subjected to Gaussian white-noise excitations, which is of the following form

$$\ddot{x}+v\dot{x}^3+u\dot{x}+(1-\alpha)x=W_1(t)+(\beta x+\gamma\dot{x})W_2(t), \tag{1}$$

where $u=(2+s-v-\alpha)$, x is the output (or called income) in the field of macroeconomics, the superscript dot represents differentiation with respect to time t. $0<\alpha\leqslant 1$ denotes the marginal propensity to consume, $s\leqslant 1$ represents the saved complementary proportion, and v describes the ratio of constant capital stock to output. Normally, only the case $v\geqslant 0$ is studied. The symbols $W_i(t)$ $(i=1,2)$ stand for stationary, independent, Gaussian white-noises processes with the properties

$$E[W_i(t)]=0,$$
$$R_{ij}(\tau)=E[W_i(t)W_j(t+\tau)]=2\pi D_{ij}\delta(\tau),$$

where $\delta(\tau)$ is Dirac delta function with $E[\bullet]$ being mean-value operator, and D_{ij} characters the intensity of the white-noises and it is assumed to be the constant as same as β, γ.

We would like to explore the long-run response of the business cycle model with the effect of random shocks. Furthermore, it is worth noting that the response is also a random process in terms of time because of the randomness of shocks. Hence, we have to search the long-run response from the view of statistical analysis. Actually, this problem can be turned out to solve the probability density function of stationary response of the random dynamical model (1). Therefore, we make a transformation for output variable x and marginal output $\dot{x}$ to figure out their relationship and their statistical properties. By setting $x = x_1$ and $\dot{x} = x_2$ and assuming them to satisfy Markov process, Eq. (1) can be rewritten as a vector form followed by the transformation, which is governed by

$$\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X}) + \boldsymbol{G}(\boldsymbol{X})\boldsymbol{W}(t), \tag{2}$$

where $\boldsymbol{X} = (x_1, x_2)^{\mathrm{T}}$ is the response vector, $\boldsymbol{W}(t) = (\boldsymbol{W}_1(t), \boldsymbol{W}_2(t))^{\mathrm{T}}$, the functions of $\boldsymbol{F}(\boldsymbol{X})$ and $\boldsymbol{G}(\boldsymbol{X})$ are all associated with response variables of $\boldsymbol{X}$ and they can be obtained from Eq. (1) as

$$\boldsymbol{F}(\boldsymbol{X}) = \begin{pmatrix} x_2 \\ -vx_2^3 - ux_2 - (1-\alpha)x_1 \end{pmatrix}, \tag{3}$$

$$\boldsymbol{G}(\boldsymbol{X}) = \begin{pmatrix} 0 & 0 \\ 1 & \beta x_1 + \gamma x_2 \end{pmatrix}. \tag{4}$$

The foregoing part will try to develop an analytical procedure to estimate the PDF response for the business cycle model.

It can be shown that the FP equation associated with the stochastic differential equation (2) is governed by

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_j}(m_j p) - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}(\sigma_{ij} p) = 0, \quad i, j = 1, 2, \tag{5}$$

where $p = p(\boldsymbol{X}, t|\boldsymbol{X}_0, t_0)$ represents the transition probability density function of the response vector $\boldsymbol{X}$ with the initial condition $p(\boldsymbol{X}, t_0|\boldsymbol{X}_0, t_0) = \delta(\boldsymbol{X} - \boldsymbol{X}_0)$. The symbols m_j and σ_{ij} are the first and second derivative moments, respectively, and they can be worked out by some mathematical manipulations. The exact expressions about them can be derived from Eq. (2), and we denote them as the vectors of $\boldsymbol{M}(\boldsymbol{X})$ and $\sigma(\boldsymbol{X})$. Here Wong–Zakai correction terms are involved in the formulas

$$\boldsymbol{M}(\boldsymbol{X}) = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -vx_2^3 - ux_2 - (1-\alpha)x_1 + \pi D_{22}\gamma(\beta x_1 + \gamma x_2) \end{pmatrix}, \tag{6}$$

$$\boldsymbol{\sigma}(\boldsymbol{X}) = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2\pi D_{11} + 2\pi D_{22}(\beta x_1 + \gamma x_2)^2 \end{pmatrix}. \tag{7}$$

Regarding Eq. (5), it is a second order differential equation. Unfortunately, with few exceptions, the exact transition PDF or unconditional PDF of this kind of equations are very difficult to be obtained due to the complication of nonlinear and random terms. According to our knowledge, only a few nonlinear equations of one-order or multi-degree-of-freedom (MDOF) equations with constant coefficients can be derived the exact close-form response solutions. Moreover, the FP equation, in general, is not amenable to exact solution in terms of the non-stationary PDF.

On the other hand, what we more concerned about is just the long-run prediction for the economical variables. Comparatively, stationary PDF response of business cycle model is already qualified to realize this purpose. Therefore, the development of accurate and efficient approximate solution procedure to get the stationary solution is desirable.

In this case, the differential equation governed the stationary PDF can be rewritten as

$$\frac{\partial}{\partial x_j}(m_j p) - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}(\sigma_{ij} p) = 0, \quad i,j = 1,2, \tag{8}$$

where m_j and σ_{ij} are the same as in Eqs. (6) and (7).

It should be reminded that the PDF of the stationary response $p(x_1,x_2)$ of the stochastic dynamical model (1) is constrained to the following conditions

$$0 \leqslant p(x_1,x_2) \leqslant 1, \quad p(x_1,x_2) \to 0 \text{ as } x_i \to \infty, \quad \iint_{\mathbf{R}^2} p(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = 1, (x_1,x_2) \in \mathbf{R}^2.$$

The EPC strategy is presented for estimating the stationary PDF response of the business cycle model. Denote $\boldsymbol{A}$ as an N-dimensional unknown parametric vector, and after that, the approximate stationary solution $\tilde{p}(\boldsymbol{X},\boldsymbol{A})$ can be assumed as

$$\tilde{p}(\boldsymbol{X},\boldsymbol{A}) = c\exp[Q(\boldsymbol{X},\boldsymbol{A})], \tag{9}$$

where c denotes the normalization constant, and $Q(\boldsymbol{X},\boldsymbol{A})$ represents a n-degree polynomial in domain $\mathbf{R}^2$ with unknown parameters a_{ij} in it. Specifically, the vector $\boldsymbol{A}$ is composed of all the unknown parameters a_{ij} and

$$Q(\boldsymbol{X},\boldsymbol{A}) = \sum_{i=1}^{n}\sum_{j=0}^{i} a_{ij} x_1^{i-j} x_2^{j}. \tag{10}$$

It is seen that the main idea of EPC strategy is to approximate the exact PDF solution by using the exponent polynomials. It is no doubt the approximation will be in better agreement with larger n-degree polynomials. Substituting Eq. (9) into Eq. (8), and using $\tilde{p}(\boldsymbol{X},\boldsymbol{A})$ to replace the exact PDF $p(x_1,x_2)$, correspondingly, the residual error between approximate solution and exact one will be occurred. We denote the residual error by $R(\boldsymbol{X},\boldsymbol{A})$ as

$$R(\boldsymbol{X},\boldsymbol{A}) = \frac{\partial m_j}{\partial x_j}\tilde{p} + m_j\frac{\partial \tilde{p}}{\partial x_j} - \frac{1}{2}\left(\frac{\partial^2 \sigma_{ij}}{\partial x_i \partial x_j}\tilde{p} + \frac{\partial \sigma_{ij}}{\partial x_j}\frac{\partial \tilde{p}}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_i}\frac{\partial \tilde{p}}{\partial x_j} + \sigma_{ij}\frac{\partial^2 \tilde{p}}{\partial x_i \partial x_j}\right). \tag{11}$$

Obviously, $R(\boldsymbol{X},\boldsymbol{A})$ should be zero if $\tilde{p}(\boldsymbol{X},\boldsymbol{A})$ is absolutely equivalent to $p(\boldsymbol{X})$ in the strong sense. However $R(\boldsymbol{X},\boldsymbol{A})$ is usually not zero since $\tilde{p}(\boldsymbol{X},\boldsymbol{A})$ is only assumed to be an approximation of $p(x_1,x_2)$ but not a definitely exact solution. Therefore, we represent the residual error as the following form to solve this problem

$$R(\boldsymbol{X},\boldsymbol{A}) = \varphi(\boldsymbol{X},\boldsymbol{A})\tilde{p}(\boldsymbol{X},\boldsymbol{A}). \tag{12}$$

Substituting m_j, σ_{ij} into Eq. (12), we get

$$\begin{aligned}\varphi(\boldsymbol{X},\boldsymbol{A}) = x_2\frac{\partial Q}{\partial x_1} - \left[3\pi\gamma D_{22}(\beta x_1+\gamma x_2)+vx_2^3+ux_2+(1-\alpha)x_1\right]\frac{\partial Q}{\partial x_2} - \\ \left[\pi D_{11}+\pi D_{22}(\beta x_1+\gamma x_2)^2\right]\left[\frac{\partial^2 Q}{\partial x_2^2}+\left(\frac{\partial Q}{\partial x_2}\right)^2\right] - \left(\pi\gamma^2 D_{22}+3vx_2^2+u\right).\end{aligned} \tag{13}$$

According to the EPC strategy, we can evaluate the unknown parametric vector $\boldsymbol{A}$ by enforcing that the residual error's projection on a set of independent functions selected properly is zero.[14] By selecting $g_k(x_1,x_2)$ as weighting function, this condition can be expressed as

$$\iint_{\mathbf{R}^2} \varphi(\boldsymbol{X},\boldsymbol{A})g_k(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0, \quad k=1,2,\cdots,N. \tag{14}$$

Moreover, taking the conditions with respect to $p(x_1,x_2)$ into account, it is very convenient to choose the weighting function as $g_k(x_1,x_2)=x_1^{k-l}x_2^l f(x_1,x_2)$ to ensure $\varphi(\boldsymbol{X},\boldsymbol{A})g_k(x_1,x_2)$ integrable in the weak sense in the span space, where $k=1,2,\cdots,N$ and $l=0,1,2,\cdots,k$, and $f(x_1,x_2)$ represents joint probability density function of the vector $\boldsymbol{X}$. Except that, numerical experience and Refs. 13–15 have shown that an effective choice for $f(x_1,x_2)$ denotes the probability density function derived from the equivalent linearization procedure subjected to Gaussian white-noise excitations. It can be formulated as

$$f(x_1,x_2) = \frac{1}{2\pi}\exp\left(-\frac{x_1^2}{2}-\frac{x_2^2}{2}\right). \tag{15}$$

Substituting Eqs. (10), (13), (15) into Eq. (14), we can get a set of nonlinear equations in terms of unknown parameter a_{ij}, and the numbers of equations are determined by the degree n of polynomial. As a result, our next task is to get the solution from this set of equations. That is to say, solving the equations to get the exact value about a_{ij} so as to determine the unknown parametric vector $\boldsymbol{A}$, which in return decide the final approximation expression of $\tilde{p}(\boldsymbol{X},\boldsymbol{A})$.

Firstly, we consider the case when $n=2$. In this case, there are five unknown parameters involved in the approximate PDF solution $\tilde{p}(\boldsymbol{X},\boldsymbol{A})$.

Correspondingly, the 2-degree polynomial can be expressed as

$$Q(\boldsymbol{X},\boldsymbol{A}) = a_1x_1+a_2x_2+a_3x_1^2+a_4x_1x_2+a_5x_2^2.$$

On the basis of the procedure introduced proviously, we need to solve five equations to determine the five unknown parameters. By reviewing the definition of Eq. (1), the economic coefficients in business cycle model have to be given within their acceptable domains $\alpha = 0.7$, $\beta = 0.5$, $\gamma = 0.5$, $u = 0.6$, $v = 0.8$, $D_{11} = 0.1$, $D_{22} = 0.4$. Finally we get the exact values of parametric vector $\boldsymbol{A} = (0.0114, 0.0033, -0.2864, -0.1156, -0.3133)$. Figure 1 shows the joint probability density of the output and its marginal output. It is seen that the PDF response obtained from the case with $n = 2$ is fulfilled as Gaussian distribution with 0.0159 of mean-value and 1.4644 of standard deviation. This result is the same as that obtained from Gaussian closure method, and that means Gaussian closure method acts as a special case of EPC method.

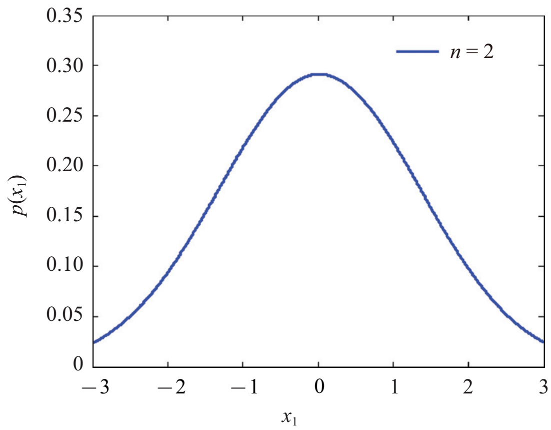

Fig. 1. Probability density function of the output in business cycle model (1).

Next, we will pay more attention to the non-Gaussian closure method and focus on considering the cases with $n = 3$ and $n = 4$. Correspondingly, there are nine unknown parameters if $n = 3$ and fourteen unknown parameters if $n = 4$ respectively. Furthermore, the number of equations needed to determine the unknown parameters will dramatically grow with the increase of degree n of polynomials.

Figure 2 shows the approximate logarithmic probability density function of output in three cases with $n = 2$, $n = 3$, and $n = 4$ respectively. Moreover, the result derived from Monte Carlo simulation is also ploted in Fig. 2, where the parameters we taken are the same as those in Fig. 1. It shows that the distributions obtained in $n = 3$ and $n = 4$ are nearly absolutely identical, which are also in good agreement with the result of Monte Carlo simulation. Numerical results further show that the results obtained from non-Gaussian closure method are much better than those from Gaussian closure especially in the tails of the PDFs.

Except that, the influences caused by random shock on the PDF solution are also discussed. The related results associated with the noise intensity are displayed in Fig. 3. It is observed that the peak of probability density function drops down monotonously with the increase of noise intensity. However, this decreasing tendency does not keep on if the noise intensity reaches a certain large value. For example, the maximum value of PDF returns to a big value once $D_{11} = 0.9$. The mathematical results appeared in Fig. 3 are quite consistent with the real economic circumstance of business cycle. D_{11} is the intensity of Gaussian white-noise. It means the stronger intensity of the random shocks coming from external factors, the more dramatic of fluctuation and evolution of business cycle. That makes it more difficult to predict and control for macroeconomic

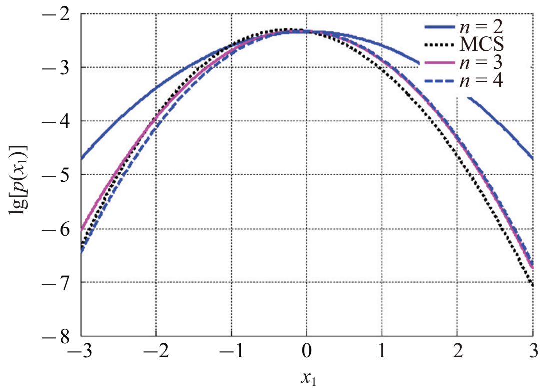

Fig. 2. Logarithmic probability density function obtained by Gaussian, non-Gaussian closure methods, and Monte Carlo simulation.

income. Therefore, decreasing the intensity of random shock is helpful to understand and predict the development of business cycle.

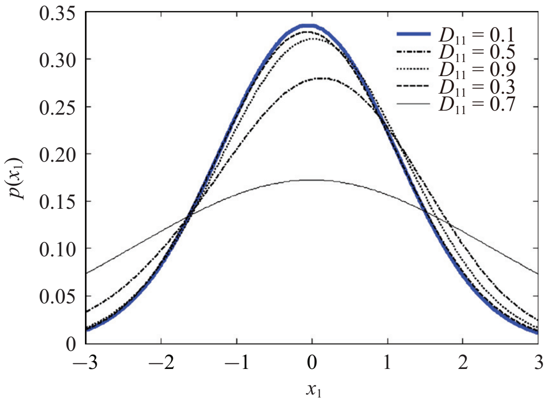

Fig. 3. Influences caused by random shock on the PDF solution.

In this paper, the stationary response of a business cycle model subjected to Gaussian white noises has been studied. An approximate solution of the associated FP equation has been pursued. Specifically, the stationary PDF of the output in business cycle model has been approximated by an exponential polynomial function with unknown parameters in it. By solving a set of integral equations with respect to those unknown parameters, the final approximate PDF is determined. Numerical results indicate that the EPC method is very applicable to get the stationary PDF for nonlinear and random excited equations, meanwhile, the results derived from EPC of non-Gaussian closure is much better than those from Gaussian closure.

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